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Quantum mechanics on the lightcone: I. the spin-zero case

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Abstract. In the usual form of quantum mechanics the wavefunctions are defined at instants of time, i.e. over constant time hyperplanes. This paper considers the wavefunctions and their operators as acting over past lightcones from points on a world line—a special case of Dirac's 'point' form of dynamics. Here the classical lightcone generators are quantized. We believe these lightcone operators, obeying the Poincaré group algebra, are new. The Hermiticity of these operators and their basis states are discussed.

1. Introduction

In the usual relativistic dynamics the position and momentum of a particle is determined as it crosses the observer's constant-time hyperplanes t_1, $t_2, \ldots$. Similarly in relativistic quantum mechanics the wavefunction is usually evaluated over a constant-time hyperplane; expectation values of observables may then be calculated as functions of t. Note that the value of the wavefunction over the hyperplane can only be known at some future time, assuming that information cannot travel faster than the speed of light. Dirac called this formalism the 'instant' form of mechanics—where the instants are the hyperplanes t.

In lightcone dynamics the particle position and momentum is determined as it crosses the stack of past lightcones T_1, $T_2, \ldots$ centred on an observer's world line. Similarly in lightcone quantum mechanics the wavefunction is evaluated on the past lightcone, and expectation values calculated accordingly. This information is, in principle, available to the observer at the origin. No observer is privileged, and provided the Poincaré algebra is satisfied, any particular observer can be selected to co-ordinate information from the others. This enables such an observer, O say, having received information from the others, to describe the state of the system under study on a past lightcone described by a particular time T_1 in O's frame of reference. If the system remains undisturbed O will be able to predict the state on past lightcones determined by O's time T where $T \geqslant T_1$. *This means that the state of the system is determined* at all points of spacetime on the 'future' side of the lightcone determined by T_1.

The approach is a special case of Dirac's 'point' form of dynamics [1]—in which the 'point' T is thought of as a past lightcone from a point on the observer's world line. Unfortunately the term 'lightcone' quantization has also been used in high-energy particle physics and string theory [3, 4] for mechanics on a hyperplane advancing at the speed of light. Dirac called this the 'front' form of mechanics, and it should not be confused with the formalism presented here.

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Since Dirac's pioneering paper, there has been relatively little work on lightcone quantum mechanics—in the sense outlined above—until Derrick's recent papers. Peres [5] obtained a set of Poincaré group operators with (undefined) inverse operators. Derrick [6] approached the problem by considering the basis states on the lightcone—we will use many of his results and notation.

Dirac's original motivation for considering quantization over alternative three-dimensional surfaces—'points' and 'fronts'—was that particle interactions may be more satisfactorily described on these surfaces. For example in electromagnetic theory, the force on a particular particle is calculated by the other particle variables on its past lightcone. Simplification may be achieved by working in lightcone coordinates from the start. We limit ourselves in this paper to the free-particle case, which is a necessary preliminary step before the more interesting interaction theory can be considered.

Section 2 introduces lightcone coordinates and the classical generators; section 3 gives the quantization procedure leading to the ten operators of the Poincaré group algebra—these are listed in (3.13); section 4 contains the proof of the commutation requirements; section 5 derives the continuity equation; section 6 considers the scalar product space and the Hermiticity of these operators; section 7 derives the basis states; section 8 is the conclusion.

2. Hamiltonian theory in lightcone coordinates

In the 'instant' form of dynamics, space translations and rotations leave the instant invariant, so the momentum and angular momentum generators will be simple compared to the energy and boost generators. In lightcone dynamics, however, a 'point' is invariant under boosts and rotations—so the boost and angular momentum (Lorentz group) generators will be simple, the energy–momentum generators more complicated.

In this section we first follow Derrick's [2] derivation of the set of Poincaré group generators (equation (2.3))—these are functions of $\boldsymbol{y}$, the position vector on the past lightcone, and its conjugate variable $\boldsymbol{\pi}'$. We then transform these generators to a new set in which the non-relativistic approximation is obvious. It is this transformed set that we will quantize.

Consider an observer whose world line we take to be at the spatial origin. Instead of labelling a particle's world line by $(t, \boldsymbol{x}(t))$, we now refer to the observer's past lightcones and relabel the particle's world line by $(T, \boldsymbol{y}(T))$, where $T_1, T_2 \ldots$ is the stack of past lightcones centred on the observer's world line. Keeping the same space components, then we determine T from t by adding to t the delay $|\boldsymbol{x}|/c$ for light to reach the observer. So

$$\begin{cases} \boldsymbol{y} = \boldsymbol{x} \\ T = t + |\boldsymbol{x}| \end{cases} \qquad \text{conversely} \qquad \begin{cases} \boldsymbol{x} = \boldsymbol{y} \\ t = T - |\boldsymbol{y}| \equiv T - y \end{cases} \tag{2.1}$$

where we have followed Derrick's notation in defining $y \equiv |\boldsymbol{y}|$ and we have used natural units where $c = 1$.

The usual action integral is $\delta[-m \int (\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2)^{1/2}] = \delta[-m \int (1 - \dot{\boldsymbol{x}}^2)^{1/2} \mathrm{d}t]$, becoming in the new coordinates (using $\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{y},\ \mathrm{d}t = \mathrm{d}T - \mathrm{d}y$)

$$\delta\left\{-m \int [(\mathrm{d}T - \mathrm{d}y)^2 - \mathrm{d}\boldsymbol{y}^2]^{1/2}\right\} = \delta\left[-m \int (1 - 2v_y + v_y^2 - \boldsymbol{v}^2)^{1/2} \mathrm{d}T\right]$$

$$\equiv \delta\left[\int L\,\mathrm{d}T\right]$$

where $\boldsymbol{v} \equiv \mathrm{d}\boldsymbol{y}/\mathrm{d}T$, $v_y \equiv \mathrm{d}y/\mathrm{d}T = \boldsymbol{v}\frac{\boldsymbol{y}}{y}$, and L is the new Lagrangian. The variable conjugate to $\boldsymbol{y}$ we call $\boldsymbol{\pi}'$, defined as

$$\boldsymbol{\pi}' \equiv \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\left[\frac{\boldsymbol{y}}{y}(1-v_y)+\boldsymbol{v}\right]}{(1-2v_y+v_y^2-v^2)^{1/2}} \simeq m\frac{\boldsymbol{y}}{y}+m\boldsymbol{v} \qquad \text{when} \quad |\boldsymbol{v}| \ll 1 \tag{2.2}$$

and referring to Derrick [2] for details the Hamiltonian can be calculated in terms of $\boldsymbol{y}$ and $\boldsymbol{\pi}'$. The complete set of Poincaré group generators is

$$\begin{aligned} &H = \frac{1}{2}y\frac{m^2+\pi'^2}{\boldsymbol{y}\cdot\boldsymbol{\pi}'} \qquad \boldsymbol{p} = \boldsymbol{\pi}' - \frac{1}{2}\boldsymbol{y}\frac{m^2+\pi'^2}{\boldsymbol{y}\cdot\boldsymbol{\pi}'} \\ &(j^{23}, j^{31}, j^{12}) \equiv \boldsymbol{J} = \boldsymbol{y}\times\boldsymbol{\pi}' \qquad (j^{10}, j^{20}, j^{30}) \equiv \boldsymbol{K} = y\boldsymbol{\pi}'\,. \end{aligned} \tag{2.3}$$

These ten generators (2.3) satisfy the Poisson bracket relations of the Poincaré group algebra which are

$$\{p^\lambda, p^\mu\} = 0 \tag{2.4}$$

$$\{j^{\lambda\mu}, p^\nu\} = \eta^{\mu\nu}p^\lambda - \eta^{\lambda\nu}p^\mu \tag{2.5}$$

$$\{j^{\lambda\mu}, j^{\nu\rho}\} = \eta^{\lambda\rho}j^{\mu\nu} + \eta^{\mu\nu}j^{\lambda\rho} - \eta^{\lambda\nu}j^{\mu\rho} - \eta^{\mu\rho}j^{\lambda\nu} \tag{2.6}$$

where the Poisson bracket is

$$\{f, g\} = \left(\frac{\partial f}{\partial \boldsymbol{y}}\right)\cdot\left(\frac{\partial g}{\partial \boldsymbol{\pi}'}\right) - \left(\frac{\partial f}{\partial \boldsymbol{\pi}'}\right)\cdot\left(\frac{\partial g}{\partial \boldsymbol{y}}\right). \tag{2.7}$$

The generators above seem to bear little resemblance to the usual ones, the reason being that for low velocities $\boldsymbol{\pi}'$ is not equal to $m\boldsymbol{v}$ but has the extra term $m\frac{\boldsymbol{y}}{y}$—see (2.2). This suggests making the canonical transformation

$$\boldsymbol{\pi}' = \boldsymbol{\pi} + m\frac{\boldsymbol{y}}{y} \tag{2.8}$$

so that for low velocities $\boldsymbol{\pi} \simeq m\boldsymbol{v}$. Substituting (2.8) into the generators (2.3) we obtain the new set

$$\begin{aligned} &H = m + \frac{1}{2}\frac{y}{\boldsymbol{y}\cdot\boldsymbol{\pi}+ym}\pi^2 \qquad \boldsymbol{p} = \boldsymbol{\pi} - \frac{1}{2}\frac{\boldsymbol{y}}{\boldsymbol{y}\cdot\boldsymbol{\pi}+ym}\pi^2 \\ &\boldsymbol{J} = \boldsymbol{y}\times\boldsymbol{\pi} \qquad \boldsymbol{K} = y\boldsymbol{\pi} + \boldsymbol{y}m\,. \end{aligned} \tag{2.9}$$

We can see that when $|\boldsymbol{\pi}| \ll m$, $H \simeq m + \frac{1}{2}\pi^2/m$, $\boldsymbol{p} \simeq \boldsymbol{\pi}$ and now we have the kinetic energy term in H as in non-relativistic theory. In the next section we will quantize this latter set of Poincaré group generators, in which the non-relativistic approximation is immediately apparent.

As $\boldsymbol{y}$ now represents the position of a particle on the observer's past lightcone, the Poisson bracket relationship $\{y^a, p^b\} = \delta^{ab}$ no longer holds ($\boldsymbol{p}$ is the generator of translations in Minkowski space, not along the lightcone). The evolution of the position operator is determined as usual:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}T} &\equiv \frac{\partial H}{\partial \boldsymbol{\pi}} = \frac{\partial}{\partial \boldsymbol{\pi}}\left(m + \frac{1}{2}\frac{y}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym}\boldsymbol{\pi}^2\right) \\ &= 2\boldsymbol{\pi}\frac{1}{2}\frac{y}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym} - \frac{1}{2}\frac{y\boldsymbol{y}}{(\boldsymbol{y}\cdot\boldsymbol{\pi} + ym)^2}\boldsymbol{\pi}^2 \\ &= \frac{y}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym}\left(\boldsymbol{\pi} - \frac{1}{2}\frac{\boldsymbol{y}}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym}\boldsymbol{\pi}^2\right) \equiv \frac{y\boldsymbol{p}}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym} \\ &\simeq \frac{\boldsymbol{p}}{m} \qquad \text{when} \quad |\boldsymbol{\pi}| \ll m\end{aligned} \tag{2.10}$$

as expected.

Note that due to the identities $\{\boldsymbol{K}, -y\} = \boldsymbol{y}$ and $\{K^a, y^b\} = -y\delta^{ab}$, which are easily verified, $\boldsymbol{y}$ is the space component of the 4-vector $y \equiv (-y, \boldsymbol{y})$ [2]—where we define a 4-vector as satisfying the Poisson bracket relationship (2.5), i.e. transforming correctly under boosts and rotations. Then we may construct the following Lorentz scalars from the two 4-vectors p and y:

$$\begin{aligned}p\cdot p &\equiv H^2 - \boldsymbol{p}^2 = m^2 \\ y\cdot y &\equiv y^2 - \boldsymbol{y}^2 = 0 \\ y\cdot p &\equiv -yH - \boldsymbol{y}\cdot\boldsymbol{p} = -(\boldsymbol{y}\cdot\boldsymbol{\pi} + ym)\,.\end{aligned} \tag{2.11}$$

(The first scale may be verified from their definitions (2.9).)

3. Quantization

In this section we quantize the ten classical generators (2.9) which must then obey the Poincaré group commutation identities

$$[p^\lambda, p^\mu] = 0 \tag{3.1}$$

$$[J^{\lambda\mu}, p^\nu] = \mathrm{i}(\eta^{\mu\nu}p^\lambda - \eta^{\lambda\nu}p^\mu) \tag{3.2}$$

$$[J^{\lambda\mu}, J^{\nu\rho}] = \mathrm{i}(\eta^{\lambda\rho}J^{\mu\nu} + \eta^{\mu\nu}J^{\lambda\rho} - \eta^{\lambda\nu}J^{\mu\rho} - \eta^{\mu\rho}J^{\lambda\nu}) \tag{3.3}$$

(cf (2.7)–(2.9)). The $\boldsymbol{y}$ and $\boldsymbol{\pi}$ now become the operators $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{\pi}}$ which we require to be Hermitian in the space $\mathcal{H}_y$, which is the Lorentz-invariant positive-definite scalar product space over the past lightcone

$$\mathcal{H}_y: \qquad \langle\phi\,|\,\psi\rangle_y \equiv \left\langle\phi\left|\frac{1}{y}\right|\psi\right\rangle \equiv \int \phi^*(\boldsymbol{y})\frac{1}{y}\psi(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\,. \tag{3.4}$$

So

$$\boldsymbol{y} \to \hat{\boldsymbol{y}} \equiv \boldsymbol{y} \qquad \boldsymbol{\pi} \to \hat{\boldsymbol{\pi}} \equiv -\mathrm{i}y^{1/2}\frac{\partial}{\partial \boldsymbol{y}}y^{-1/2}\,. \tag{3.5}$$

Note that the $y^{1/2}$ terms in the latter are necessary for $\hat{\pi}$ to be a self-adjoint operator with respect to the scalar product $\mathcal{H}_y$. The relatively simple boost and rotation generators from (2.9) become

$$\boldsymbol{J} \equiv (J^{23}, J^{31}, J^{12}) = \boldsymbol{y} \times \hat{\boldsymbol{\pi}} = -\mathrm{i}\boldsymbol{y} \times \frac{\partial}{\partial \boldsymbol{y}} \tag{3.6}$$

$$\boldsymbol{K} \equiv (J^{10}, J^{20}, J^{30}) = y^{1/2}\, \hat{\boldsymbol{\pi}}\, y^{1/2} + \boldsymbol{y}m = -\mathrm{i}y \frac{\partial}{\partial \boldsymbol{y}} + \boldsymbol{y}m\,. \tag{3.7}$$

These operators are Hermitian in $\mathcal{H}_y$ by inspection.

In the case of the H and $\boldsymbol{p}$ operators (2.9) there is an awkward $1/(\boldsymbol{y}\cdot\boldsymbol{\pi} + ym)$ component which appears difficult to quantize. We first form the Hermitian operator corresponding to $(\boldsymbol{y}\cdot\boldsymbol{\pi} + ym)$ which we will call Σ_m:

$$\boldsymbol{y}\cdot\boldsymbol{\pi} + ym \to \Sigma_m \equiv \hat{\boldsymbol{y}} \circ \hat{\boldsymbol{\pi}} + ym$$

where $A \circ B$ is the symmetric product $\frac{1}{2}(AB + BA)$,

$$\begin{aligned} \Sigma_m &= \hat{\boldsymbol{y}} \circ -\mathrm{i}y^{1/2}\frac{\partial}{\partial \boldsymbol{y}}y^{-1/2} + ym \\ &= -\mathrm{i}\left(\boldsymbol{y}\cdot\frac{\partial}{\partial \boldsymbol{y}} + 1\right) + ym \\ &= -\mathrm{i}\frac{\partial}{\partial y}y + ym \end{aligned} \tag{3.8}$$

which is Hermitian due to its symmetric construction. We expect Σ_m to be Lorentz-invariant as is its counterpart in classical theory (see (2.11)), and it may be verified that Σ_m does in fact commute with all $\boldsymbol{J}$, $\boldsymbol{K}$.

Quantizing the $1/(\boldsymbol{y}\cdot\boldsymbol{\pi} + ym)$ term, this must be an inverse operator to Σ_m which we will call Σ_m^{-1}. We define Σ_m^{-1} as

$$\Sigma_m^{-1} f(\boldsymbol{y}) = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}my}}{y}\int_0^y \mathrm{e}^{\mathrm{i}my'} f(y', \theta, \phi)\,\mathrm{d}y' \tag{3.9a}$$

or equivalently

$$\Sigma_m^{-1} f(\boldsymbol{y}) = \mathrm{i}\int_0^1 \mathrm{e}^{\mathrm{i}my(\alpha-1)} f(\alpha\boldsymbol{y})\,\mathrm{d}\alpha\,. \tag{3.9b}$$

Then $\Sigma_m^{-1}\Sigma_m f = \Sigma_m \Sigma_m^{-1} f = f$ as required (see appendix 1). We can see from (3.9) that $\Sigma_m^{-1} f(\boldsymbol{y})$ depends on the values of f along the line between the point $\boldsymbol{y}$ and the vertex of the lightcone. The Hermiticity of Σ_m^{-1} follows from the Hermiticity of Σ_m.

Now we can quantize the H generator from (2.9):

$$\begin{aligned} H &\equiv m + \frac{1}{2}\left(\frac{1}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym}\right) y\boldsymbol{\pi}^2 \\ &\to\ m + \frac{1}{2}\Sigma_m^{-1} y^{1/2}\hat{\boldsymbol{\pi}}^2 y^{1/2} = m - \frac{1}{2}\Sigma_m^{-1} y\nabla^2 \end{aligned} \tag{3.10a}$$

where ∇^2 stands for $|\partial/\partial y|^2$. It is of interest to determine the non-relativistic approximation to (3.10*a*). To this end we will temporarily reinsert the speed of light c into (3.10*a*) (we have been using natural units with $c = 1$) obtaining

$$\frac{H}{c} = mc - \frac{1}{2}\Sigma_{mc}^{-1}\, y\nabla^2 \,. \tag{3.10b}$$

But $\Sigma_{mc} \equiv -\mathrm{i}\frac{\partial}{\partial y}y + ycm \simeq ymc$ as $c \to \infty$, which is the non-relativistic approximation—when the past lightcones get flattened out to become constant-time hyperplanes. Then we may infer that $\Sigma_{mc}^{-1} \simeq \frac{1}{ymc}$ as $c \to \infty$, and inserting this into (3.10*b*)

$$\frac{H}{c} \simeq mc - \frac{1}{2}\frac{1}{ymc}y\nabla^2 = mc - \frac{1}{2mc}\nabla^2 \qquad \text{as} \quad c \to \infty \,.$$

Taking out the rest energy mc^2, we recover the non-relativistic Schrödinger equation $H = -\frac{1}{2m}\nabla^2$.

From (3.10*a*) we can see that the new evolution equation is

$$\mathrm{i}\partial_T\psi = H\,\psi \equiv m\psi - \tfrac{1}{2}\Sigma_m^{-1}y\nabla^2\,\psi \tag{3.11}$$

where ψ is a scalar. Note however that (3.11) is of first order in time as compared to the usual spin-zero relativistic equation—the Klein–Gordon—which is of second order. So if the wavefunction is given on the lightcone, then its evolution is uniquely defined. In section 5 we obtain the positive and negative energy eigenfunctions of the Hamiltonian operator (3.10*a*).

To quantize the p generator we use the commutation requirement from (3.2), namely $[\boldsymbol{K}, H] = \mathrm{i}p$. So recalling that $\boldsymbol{K}$ commutes with Σ_m and hence with Σ_m^{-1},

$$\begin{aligned} p = -\mathrm{i}[\boldsymbol{K}, H] &\equiv -\mathrm{i}[\boldsymbol{K}, m - \Sigma_m^{-1}\tfrac{1}{2}y\nabla^2] \\ &= \frac{\mathrm{i}}{2}\Sigma_m^{-1}[\boldsymbol{K}, y\nabla^2] \equiv \frac{\mathrm{i}}{2}\Sigma_m^{-1}\left[-\mathrm{i}y\frac{\partial}{\partial y} + my,\, y\nabla^2\right] \\ &= \frac{\mathrm{i}}{2}\Sigma_m^{-1}\left(-\left[\mathrm{i}y\frac{\partial}{\partial y},\, y\right]\nabla^2 - \mathrm{i}y\left[y,\, \nabla^2\right]\frac{\partial}{\partial y} + my\left[y,\, \nabla^2\right]\right) . \end{aligned}$$

We now use $[y, \nabla^2] = -\frac{2}{y}\frac{\partial}{\partial y}y$, which we derive in appendix 3. Then

$$\begin{aligned} p = \frac{\mathrm{i}}{2}\Sigma_m^{-1}&\left(-\mathrm{i}y\nabla^2 - \mathrm{i}y\left(-\frac{2}{y}\frac{\partial}{\partial y}y\right)\frac{\partial}{\partial y} - 2my\frac{\partial}{\partial y}\right) \\ &= \frac{\mathrm{i}}{2}\Sigma_m^{-1}\left(-\mathrm{i}y\nabla^2 - 2\Sigma_m\frac{\partial}{\partial y}\right) \qquad \text{(recalling (3.8))} \\ &= -\mathrm{i}\frac{\partial}{\partial y} + \frac{1}{2}\Sigma_m^{-1}y\nabla^2 \,. \end{aligned} \tag{3.12a}$$

Inserting the speed of light c, (3.12*a*) becomes

$$p = -\mathrm{i}\frac{\partial}{\partial y} + \frac{1}{2}\Sigma_{mc}^{-1}\,y\nabla^2 \simeq -\mathrm{i}\frac{\partial}{\partial y} + \frac{1}{2}\frac{1}{ymc}y\nabla^2 \simeq -\mathrm{i}\frac{\partial}{\partial y} \qquad \text{as} \quad c \to \infty \,. \tag{3.12b}$$

We have now defined in (3.6), (3.7), (3.10*a*) and (3.12*a*) all the ten operators satisfying the Poincaré group algebra which we collect below:

$$H = m - \frac{1}{2}\Sigma_m^{-1} y \nabla^2 \qquad \boldsymbol{p} = -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}} + \frac{1}{2}\Sigma_m^{-1} \boldsymbol{y} \nabla^2$$
$$\boldsymbol{J} = -\mathrm{i}\boldsymbol{y} \times \frac{\partial}{\partial \boldsymbol{y}} \qquad \boldsymbol{K} = -\mathrm{i}y\frac{\partial}{\partial \boldsymbol{y}} + \boldsymbol{y}m\,. \tag{3.13}$$

There is no problem in taking the zero-mass limit of these operators which then become

$$H = -\frac{1}{2}\Sigma_0^{-1} y \nabla^2 \qquad \boldsymbol{p} = -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}} + \frac{1}{2}\Sigma_0^{-1} \boldsymbol{y} \nabla^2 \qquad \boldsymbol{K} = -\mathrm{i}y\frac{\partial}{\partial \boldsymbol{y}}\,. \tag{3.14}$$

In the next section we will show that the above operators do in fact obey the Poincaré group commutation identities (3.1)–(3.3), in particular the more difficult of these identities, namely the commutation of the energy–momentum operators (3.1), i.e. $[p^\lambda,\ p^\mu] = 0$.

4. Commutation relations

An important property of the operators Σ_m and Σ_m^{-1} is that they are functions of y and so commute with any function of the angular variables $\frac{\boldsymbol{y}}{y}$. The commutator requirement $[K^a, p^b] = \mathrm{i}\delta^{ab} H$ may be checked using the identities $[K^a, -\mathrm{i}\partial/\partial y^b] = (y^b/y)\partial/\partial y^a + \mathrm{i}m\delta^{ab}$, $[K^a, y^b] = -\mathrm{i}y\delta^{ab}$, which are easily verified, and $[K^a, \nabla^2] = -(2/y)\Sigma_m \partial/\partial y^a$, which we derive in appendix 3. Next we calculate $[H, \boldsymbol{p}]$, for which we need the identity

$$\left[\nabla^2, \frac{\boldsymbol{y}}{y}\right] = 2\mathrm{i}\left[\frac{\partial}{\partial \boldsymbol{y}}, \frac{1}{y}\Sigma_m\right] \tag{4.1}$$

which we derive in the appendix 3—equation (A3.7). Then

$$\begin{aligned}
[H, \boldsymbol{p}] &= \left[m - \Sigma_m^{-1}\frac{1}{2}y\nabla^2, -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}} + \frac{1}{2}\frac{\boldsymbol{y}}{y}\Sigma_m^{-1} y \nabla^2\right] \\
&= \frac{\mathrm{i}}{2}\left[\Sigma_m^{-1} y, \frac{\partial}{\partial \boldsymbol{y}}\right]\nabla^2 - \frac{1}{4}\Sigma_m^{-1} y \left[\nabla^2, \frac{\boldsymbol{y}}{y}\right]\Sigma_m^{-1} y \nabla^2 \\
&= \frac{\mathrm{i}}{2}\left[\Sigma_m^{-1} y, \frac{\partial}{\partial \boldsymbol{y}}\right]\nabla^2 - \frac{\mathrm{i}}{2}\Sigma_m^{-1} y \left[\frac{\partial}{\partial \boldsymbol{y}}, \frac{1}{y}\Sigma_m\right]\Sigma_m^{-1} y \nabla^2 \qquad \text{(using (4.1))} \\
&= 0 \qquad \text{as desired.}
\end{aligned} \tag{4.2}$$

Similarly the components of $\boldsymbol{p}$ may be shown to commute, again using (4.1), and the remaining Poincaré group commutator identities are easily verified.

5. The continuity equation

From (2.6) we can derive a continuity equation in the form

$$\frac{\partial}{\partial T}\rho^0 + \frac{\partial}{\partial \boldsymbol{y}} \cdot \boldsymbol{\rho} = 0 \tag{5.1}$$

where

$$\rho^0 = \frac{1}{2}\left[\psi^* \frac{1}{y}\Sigma_m \psi + (\Sigma_m \psi)^* \frac{1}{y}\psi\right] \tag{5.2}$$

$$\boldsymbol{\rho} = \frac{1}{2}\left(\psi^* \boldsymbol{p}\psi + (\boldsymbol{p}\psi)^* \psi\right) \equiv \frac{1}{2}\left(-\psi^* \,\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\psi + \psi^* \frac{1}{2}\frac{\boldsymbol{y}}{y}\Sigma_m^{-1} y \nabla^2 \psi + \mathrm{CC}\right) \tag{5.3}$$

and CC stands for the complex conjugate terms. Recall that in the non-relativistic limit $\boldsymbol{p} \simeq -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}$ and $\Sigma_m \simeq ym$ in natural units. Then $\rho^0 \simeq m\psi^* \psi$ and

$$\boldsymbol{\rho} \simeq \frac{1}{2}\left[-\psi^* \,\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\psi - \left(\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\psi\right)^* \psi\right].$$

To show the continuity equation is satisfied, we first calculate $\frac{\partial}{\partial \boldsymbol{y}} \cdot \boldsymbol{\rho}$:

$$\begin{aligned}
2\left(\frac{\partial}{\partial \boldsymbol{y}} \cdot \boldsymbol{\rho}\right) &= -\frac{\partial}{\partial \boldsymbol{y}}\psi^* \cdot \mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\psi - \psi^* \,\mathrm{i}\nabla^2\psi + \frac{\partial}{\partial \boldsymbol{y}}\psi^* \cdot \frac{1}{2}\frac{\boldsymbol{y}}{y}\Sigma_m^{-1} y\nabla^2\psi \\
&\quad + \psi^* \frac{1}{2}\frac{\partial}{\partial \boldsymbol{y}} \cdot \frac{\boldsymbol{y}}{y}\Sigma_m^{-1} y \nabla^2 \psi + \mathrm{CC} \\
&= -\frac{\partial}{\partial \boldsymbol{y}}\psi^* \cdot \mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\psi - \psi^* \mathrm{i}\nabla^2\psi \\
&\quad + \frac{1}{2}\left(\frac{\partial}{\partial y} + \frac{1}{y}\right)\psi^* \,\Sigma_m^{-1} y\nabla^2\psi + \frac{1}{2}\psi^* \left(\frac{\partial}{\partial y} + \frac{1}{y}\right)\Sigma_m^{-1} y \nabla^2\psi + \mathrm{CC}.
\end{aligned}$$

The first term is cancelled by its complex conjugate, and we now use

$$\left(\frac{\partial}{\partial y} + \frac{1}{y}\right) = \frac{1}{y}\frac{\partial}{\partial y} y = \left(\frac{\mathrm{i}}{y}\Sigma_m - \mathrm{i}m\right)$$

from (3.8) to obtain

$$\begin{aligned}
2\left(\frac{\partial}{\partial \boldsymbol{y}} \cdot \boldsymbol{\rho}\right) &= -\psi^* \,\mathrm{i}\nabla^2\psi + \frac{1}{2}\left[\left(\frac{\mathrm{i}}{y}\Sigma_m - \mathrm{i}m\right)\psi\right]^* \Sigma_m^{-1} y \nabla^2 \psi \\
&\quad + \frac{1}{2}\psi^* \left(\frac{\mathrm{i}}{y}\Sigma_m - \mathrm{i}m\right)\Sigma_m^{-1} y\nabla^2\psi + \mathrm{CC} \\
&= -\frac{\mathrm{i}}{2}\psi^* \,\nabla^2\psi + \frac{1}{2}\left(\frac{\mathrm{i}}{y}\Sigma_m \psi\right)^* \Sigma_m^{-1} y \nabla^2\psi + \mathrm{CC}
\end{aligned} \tag{5.4}$$

whereas

$$2\left(\frac{\partial}{\partial T}\rho^0\right) = \left(\frac{\partial}{\partial T}\psi\right)^* \frac{1}{y}\Sigma_m \psi + \psi^* \frac{1}{y}\Sigma_m \frac{\partial}{\partial T}\psi + \mathrm{CC}.$$

If we now insert the evolution equation (3.11) we obtain

$$2\left(\frac{\partial}{\partial T}\rho^0\right) = \left[\left(-\mathrm{i}m + \tfrac{\mathrm{i}}{2}\Sigma_m^{-1}y\nabla^2\right)\psi\right]^* \frac{1}{y}\Sigma_m\psi + \psi^* \frac{1}{y}\Sigma_m\left(-\mathrm{i}m + \tfrac{\mathrm{i}}{2}\Sigma_m^{-1}y\nabla^2\right)\psi + \mathrm{cc}$$
$$= \left(\tfrac{\mathrm{i}}{2}\Sigma_m^{-1}y\nabla^2\psi\right)^* \frac{1}{y}\Sigma_m\psi + \frac{\mathrm{i}}{2}\psi^*\nabla^2\psi + \mathrm{cc}. \tag{5.5}$$

Adding (5.4) and (5.5) results in all terms cancelling when the complex conjugates are included.

6. Scalar products

Recall the Hamiltonian operator $H = m - \frac{1}{2}\Sigma_m^{-1}y\nabla^2$. The factors Σ_m^{-1} and $y\nabla^2$ are each Hermitian in the positive-definite scalar product space $\mathcal{H}_y$ from (3.4). However H is not Hermitian in $\mathcal{H}_y$ as the two factors do not commute. We will show that H is Hermitian in the following (indefinite) inner product space

$$\mathcal{H}_{\mathrm{spin\ zero}} \equiv \mathcal{H}_{\mathrm{SZ}}: \qquad \langle\phi\,|\,\psi\rangle_{\mathrm{SZ}} = \frac{1}{2}\int\frac{\mathrm{d}^3\boldsymbol{y}}{y}\,[(\Sigma_m\phi)^*\psi + \phi^*\Sigma_m\psi] \equiv \langle\phi\,|\,\Sigma_m\psi\rangle_y\,. \tag{6.1}$$

It may be wondered why we are now introducing the new space $\mathcal{H}_{\mathrm{SZ}}$ (with all the problems of its indefinite metric) after quantizing consistently in $\mathcal{H}_y$. This step is forced on us as any attempt to symmetrize the factors of H to obtain a Hamiltonian Hermitian in $\mathcal{H}_y$ fails, because the resulting operators do not commute as required. The lack of a positive-definite scalar product in the spin-zero case is reflected in the usual spin-zero relativistic (Klein–Gordon) theory, where the indefinite inner product space is

$$\mathcal{H}_{\mathrm{KG}}: \qquad \langle\phi\,|\,\psi\rangle_{\mathrm{KG}} = \frac{\mathrm{i}}{2}\int\mathrm{d}^3\boldsymbol{x}\left[\phi^*\frac{\partial}{\partial t}\psi - \left(\frac{\partial}{\partial t}\phi^*\right)\psi\right].$$

However the introduction of $\mathcal{H}_{\mathrm{SZ}}$ is consistent with the results of the last section, in that $\langle\psi\,|\,\psi\rangle_{\mathrm{SZ}} = \int\rho^0\,\mathrm{d}^3\boldsymbol{y}$ where ρ^0 is the conserved (indefinite) density of (5.2).

The Hamiltonian operator may be shown to be self-adjoint in $\mathcal{H}_{\mathrm{SZ}}$ as follows. Considering the operator $2(m - H) \equiv \Sigma_m^{-1}y\nabla^2$

$$2\langle\phi\,|\,(m-H)\psi\rangle_{\mathrm{SZ}} \equiv 2\langle\phi\,|\,\Sigma_m(m-H)\psi\rangle_y \equiv \langle\phi\,|\,\Sigma_m\Sigma_m^{-1}y\nabla^2\psi\rangle_y$$
$$= \langle\phi\,|\,y\nabla^2\psi\rangle_y = \langle y\nabla^2\phi\,|\,\psi\rangle_y = \langle\Sigma_m(\Sigma_m^{-1}y\nabla^2)\phi\,|\,\psi\rangle_y$$
$$= 2\langle(m-H)\phi\,|\,\Sigma_m\psi\rangle_y \equiv 2\langle(m-H)\phi\,|\,\Sigma_m\psi\rangle_{\mathrm{SZ}}\,.$$

The rotation-boost operators are self-adjoint with respect to $\mathcal{H}_{\mathrm{SZ}}$, as they commute with Σ_m and are self-adjoint with respect to $\mathcal{H}_y$. Then as H is self-adjoint, so is $p \equiv -\mathrm{i}[K, H]$.

7. The basis states on the lightcone

The following are positive- and negative-energy eigensolutions of the Hamiltonian H:

$$u_{\boldsymbol{k}} \equiv \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \qquad v_{\boldsymbol{k}} \equiv \mathrm{e}^{-\mathrm{i}\omega s - \mathrm{i}my - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \tag{7.1}$$

where $\omega = +\sqrt{m^2 + \boldsymbol{k}^2}$. Note that the wavefunctions introduced above are improper in that they are not normalizable, but they have no singularities. The proof that

$$H\, u_{\boldsymbol{k}} = \omega\, u_{\boldsymbol{k}} \tag{7.2}$$

we leave to appendix 2.

We could also prove that $H v_{\boldsymbol{k}} = -\omega v_{\boldsymbol{k}}$ by direct calculation but it is instructive to use the following method. First note from (3.8) the identity $(\Sigma_{(-m)})^* = -\Sigma_m$ implying $(\Sigma^{-1}_{(-m)})^* = -\Sigma^{-1}_m$. Then if in (7.2) we replace m by $-m$, take the complex conjugate of the result, multiply each side by -1 and use the identity above, we obtain

$$H v_{\boldsymbol{k}} = -\omega v_{\boldsymbol{k}}\,. \tag{7.3}$$

We emphasize that both positive- and negative-energy solutions obey $\mathrm{i}\frac{\partial}{\partial T}\psi = H\psi$. This is different from the usual relativistic spin-zero case, where two first-order evolution equations are needed:

$$\mathrm{i}\frac{\partial}{\partial t}\psi = \hat{\omega}\psi \qquad \text{positive energy solutions}$$

$$\mathrm{i}\frac{\partial}{\partial t}\psi = -\hat{\omega}\psi \qquad \text{negative energy solutions}$$

where $\hat{\omega} \equiv (m^2 - \nabla^2)^{1/2}$.

8. Conclusion

We have defined the p and J operators (3.13), and proved that they satisfy the Poincaré group algebra. The operators are Hermitian in a Lorentz-invariant (indefinite) scalar product space—in this respect being similar to the spin-zero relativistic case. Our basis states are similar to those found by Derrick in [6] , but have the extra factor $\mathrm{e}^{-\mathrm{i}my}$.

The main reason for Dirac to have [1] considered alternative forms of dynamics is that interactions may be more readily introduced. Any two points on a past lightcone are in general spacelike-separated, unless they are on a common ray from the vertex in which case they are lightlike-separated. However we suggest that interactions on a past lightcone would not violate causality because it is only in the future (at the vertex of the lightcone) that these interactions may be observed.

Our next paper [7] deals with the spin-$\frac{1}{2}$ case in lightcone coordinates—here the scalar product space is the positive-definite space $\mathcal{H}_y$.

Appendix 1. The operators Σ_m and Σ_m^{-1}

Recall from (3.8) that

$$\Sigma_m \equiv -\mathrm{i}\frac{\partial}{\partial y}y + ym = \mathrm{e}^{-\mathrm{i}my}\left(-\mathrm{i}\frac{\partial}{\partial y}y\right)\mathrm{e}^{\mathrm{i}my} \equiv \mathrm{e}^{-\mathrm{i}my}\,\Sigma_0\,\mathrm{e}^{\mathrm{i}my}\,.$$

(Our operator $\Sigma_0 \equiv -\mathrm{i}\frac{\partial}{\partial y}y$ is the same as Derrick's operator Σ [6]). We first find the inverse of Σ_0 as follows:

$$\begin{aligned} &\Sigma_0\Sigma_0^{-1}f = f \qquad -\mathrm{i}\frac{\partial}{\partial y}y\Sigma_0^{-1}f = f \\ &\frac{\partial}{\partial y}\left(y\Sigma_0^{-1}f\right) = \mathrm{i}f \qquad \Sigma_0^{-1}f = \frac{\mathrm{i}}{y}\int_0^y f(y',\theta,\phi)\,\mathrm{d}s' + \frac{K}{y} \end{aligned} \tag{A1.1}$$

where K is any function of the angular variables. We require $\Sigma_0^{-1}f$ to be regular at $y = 0$ so we must have $K = 0$—then Σ_0^{-1} is a *linear* operator. Note that f must be regular for $\Sigma_0^{-1}f$ to be defined; for example if f is the improper function $f = 1/y$, then $\Sigma_0 f = 0$ and $\Sigma_0^{-1}f = \infty$, as would be expected. For many purposes it is convenient to make the following substitution into the integral above:

$$y' = y\alpha \qquad \mathrm{d}y' = y\,\mathrm{d}\alpha$$

leading to

$$\Sigma_0^{-1}f = \mathrm{i}\int_0^1 f(\alpha y)\,\mathrm{d}\alpha\,. \tag{A1.2}$$

We have shown $\Sigma_0\Sigma_0^{-1}f = f$, now we need to prove that $\Sigma_0^{-1}\Sigma_0 f = f$:

$$\begin{aligned} \Sigma_0^{-1}\Sigma_0 f &\equiv \int_0^1\left(\boldsymbol{y}\cdot\frac{\partial}{\partial\boldsymbol{y}} + 1\right) f(\alpha y_1,\alpha y_2,\alpha y_3)\,\mathrm{d}\alpha \\ &= \int_0^1\left(\alpha\frac{\partial}{\partial\alpha} + 1\right) f(\alpha y_1,\alpha y_2,\alpha y_3)\,\mathrm{d}\alpha \\ &= \int_0^1 \frac{\partial}{\partial\alpha}\left(\alpha f(\alpha y_1,\alpha y_2,\alpha y_3)\right)\mathrm{d}\alpha = \left|_0^1 \alpha f(\alpha y_1,\alpha y_2,\alpha y_3)\right| \\ &= f(\boldsymbol{y}) \equiv f\,. \end{aligned}$$

As $\Sigma_m \equiv \mathrm{e}^{-\mathrm{i}my}\,\Sigma_0\,\mathrm{e}^{\mathrm{i}my}$ then $\Sigma_m^{-1} \equiv \mathrm{e}^{-\mathrm{i}my}\,\Sigma_0^{-1}\,\mathrm{e}^{\mathrm{i}my}$. So using (A1.1) or (A1.2) we arrive at the definition

$$\begin{aligned} \Sigma_m^{-1}f(\boldsymbol{y}) &= \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}my}}{y}\int_0^y \mathrm{e}^{\mathrm{i}my'} f(y',\theta,\phi)\,\mathrm{d}y' \\ \text{or equivalently}\qquad & \mathrm{i}\int_0^1 \mathrm{e}^{\mathrm{i}my(\alpha-1)} f(\alpha\boldsymbol{y})\,\mathrm{d}\alpha \end{aligned} \tag{A1.3}$$

which is (3.9).

Appendix 2. The basis state u_k

Proof of $H\, u_k = \omega\, u_k$:

$$H\, u_k \equiv (m - \Sigma_m^{-1}\frac{1}{2}y\nabla^2)\, e^{i\omega y - imy + i\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} - \frac{1}{2}\Sigma_m^{-1} y\Big[-\boldsymbol{k}^2 - (\omega-m)^2 + \frac{2i(\omega-m)}{y} + 2i\boldsymbol{k}\cdot i(\omega-m)\frac{\boldsymbol{y}}{y}\Big] e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + \frac{1}{2}\Sigma_m^{-1} y\Big[2\omega^2 - 2\omega m - \frac{2i(\omega-m)}{y} + 2(\omega-m)\boldsymbol{k}\cdot\frac{\boldsymbol{y}}{y}\Big] e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + (\omega-m)\Sigma_m^{-1}(\omega y + \boldsymbol{k}\cdot\boldsymbol{y} - i)\, e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$\equiv m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + i(\omega-m)\int_0^1 e^{imy(\alpha-1)}\,(\omega\alpha y + \boldsymbol{k}\cdot\alpha\boldsymbol{y} - i)\, e^{+i(\omega-m)\alpha y + i\boldsymbol{k}\cdot\alpha\boldsymbol{y}}\, d\alpha$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + i(\omega-m)e^{-imy}\int_0^1 \frac{\partial}{\partial\alpha}\left(-i\alpha e^{i\omega\alpha y + i\boldsymbol{k}\cdot\alpha\boldsymbol{y}}\right) d\alpha$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + i(\omega-m)e^{-imy}\,\Big|_0^1 - i\alpha e^{i\omega\alpha y + i\boldsymbol{k}\cdot\alpha\boldsymbol{y}}\Big|$$

$$= m e^{i(\omega-m)y+i\boldsymbol{k}\cdot\boldsymbol{y}} + (\omega-m)\, e^{-imy} e^{i\omega y + i\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= \omega\, e^{i\omega y - imy + i\boldsymbol{k}\cdot\boldsymbol{y}} = \omega\, u_k \qquad \text{(A2.1)}$$

as desired.

Appendix 3. Identities involving Σ_m, Σ_m^{-1}, ∇^2

First note the following commutation identities with ∇^2:

$$[y, \nabla^2] \equiv \left[y, \frac{\partial^2}{\partial y^2}\right] = -\frac{2}{y}\frac{\partial}{\partial y}y \qquad \text{(A3.1)}$$

$$\left[\frac{1}{y}, \nabla^2\right] = \left[\frac{1}{y}, \frac{1}{y}\frac{\partial}{\partial y}\frac{\partial}{\partial y}y\right] = \frac{2}{y^2}\frac{\partial}{\partial y} \qquad \text{(A3.2)}$$

$$\left[\frac{\boldsymbol{y}}{y}, \nabla^2\right] = \boldsymbol{y}\left[\frac{1}{y}, \nabla^2\right] + [\boldsymbol{y}, \nabla^2]\frac{1}{y} = 2\frac{\boldsymbol{y}}{y^2}\frac{\partial}{\partial y} - 2\frac{\partial}{\partial \boldsymbol{y}}\frac{1}{y} \qquad \text{(A3.3)}$$

and

$$[\boldsymbol{K}, \nabla^2] \equiv \left[-iy\frac{\partial}{\partial \boldsymbol{y}} + \boldsymbol{y}m, \nabla^2\right] = -i\,[y, \nabla^2]\,\frac{\partial}{\partial \boldsymbol{y}} + [\boldsymbol{y}, \nabla^2]\, m$$

$$= -i\left(-\frac{2}{y}\frac{\partial}{\partial y}y\right)\frac{\partial}{\partial \boldsymbol{y}} + \left(2\frac{\partial}{\partial \boldsymbol{y}}\right)m = -\frac{2}{y}\left(-i\frac{\partial}{\partial y}y + ym\right)\frac{\partial}{\partial \boldsymbol{y}}$$

$$= -\frac{2}{y}\Sigma_m\frac{\partial}{\partial \boldsymbol{y}}. \qquad \text{(A3.4)}$$

Also

$$\left[\Sigma_m, \frac{\partial}{\partial y^a}\right] \equiv \left[-\mathrm{i}\left(\boldsymbol{y}\cdot\frac{\partial}{\partial \boldsymbol{y}}+1\right)+ym,\ \frac{\partial}{\partial y^a}\right] = \mathrm{i}\frac{\partial}{\partial y^a} - \frac{y^a}{y}m\,. \tag{A3.5}$$

So

$$\left[\frac{1}{y}\Sigma_m, \frac{\partial}{\partial y^a}\right] = \left[\frac{1}{y}, \frac{\partial}{\partial y^a}\right]\Sigma_m + \frac{1}{y}\left[\Sigma_m, \frac{\partial}{\partial y^a}\right] = \frac{y^a}{y^3}\Sigma_m + \frac{1}{y}\left(\mathrm{i}\frac{\partial}{\partial y^a} - \frac{y^a}{y}m\right)$$
$$= -\mathrm{i}\frac{y^a}{y^3}\frac{\partial}{\partial y}y + \frac{1}{y}\left(\mathrm{i}\frac{\partial}{\partial y^a}\right) = -\mathrm{i}\frac{y^a}{y^2}\frac{\partial}{\partial y} + \mathrm{i}\frac{\partial}{\partial y^a}\frac{1}{y}\,. \tag{A3.6}$$

Comparison of (A3.6) with (A3.3) shows that

$$\left[\frac{\boldsymbol{y}}{y}, \nabla^2\right] = 2\mathrm{i}\left[\frac{1}{y}\Sigma_m, \frac{\partial}{\partial \boldsymbol{y}}\right]\,. \tag{A3.7}$$

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